# Electricity and Magnetism, Exam 2, 09/03/2018
# - with solutions -

5 questions

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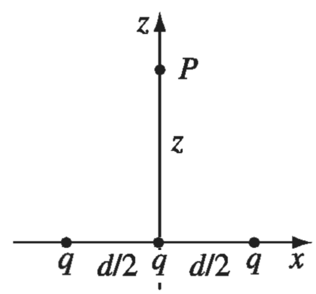
| Write your name and student number on the answer sheet. Use of a calculator is allowed. You may make use of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\mathbf{A}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector in the x-direction, and $T$ is a scalar. |

1. *15 points.* Consider three equal charges ($q$), a distance $d/2$ apart.




   (a) Find the electric field a distance $z$ above the midpoint of the three equal charges.
   *Make use of the superposition principle. For two charges, see Example 2.1:*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{[z^2 + (d/2)^2]^{3/2}}\hat{\mathbf{z}}.$$

   *Now we have to add the electric field of the third charge, which is*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}.$$

   *The answer is the sum of the two terms:*

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}[\frac{2z}{[z^2 + (d/2)^2]^{3/2}} + \frac{1}{z^2}]\hat{\mathbf{z}}.$$

   (b) What is the energy required to assemble this charge configuration?
   *First charge requires no work to be done:* $W_1 = 0$ *For the second, it is equal to the potential:*

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d/2}.$$

   *For the last charge, we have*

$$W_3 = \frac{1}{4\pi\epsilon_0}[\frac{q^2}{d/2} + \frac{q^2}{d}].$$

*The total energy is the sum of these three:*

$$W_{total} = W_1 + W_2 + W_3 = \frac{1}{4\pi\epsilon_0}[\frac{2q^2}{d/2} + \frac{q^2}{d}] = \frac{5q^2}{4\pi\epsilon_0 d}$$

(c) Consider a configuration where the central charge is negative, and the two outer ones are positive. What is the total energy required to assemble this configuration?
*$W_1$ is still zero, $W_2$ is negative:*

$$W_2 = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{d/2},$$

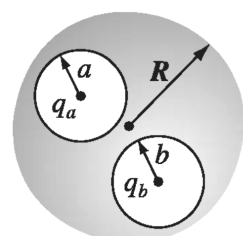
*and $W_3$ has a positive and negative part:*

$$W_3 = \frac{1}{4\pi\epsilon_0}[\frac{-q^2}{d/2} + \frac{q^2}{d}].$$

*The sum of the three terms is now*

$$W_{total} = W_1 + W_2 + W_3 = \frac{1}{4\pi\epsilon_0}[\frac{-2q^2}{d/2} + \frac{q^2}{d}] = \frac{-3q^2}{4\pi\epsilon_0 d}$$

2. *20 points.* Two spherical cavities, of radii $a$ and $b$, are hollowed out from the interior of a (neutral) conducting solid sphere of radius $R$. At the center of each cavity a point charge is placed - call these charges $q_a$ and $q_b$.




(a) Find the surface charge densities $\sigma_a$, $\sigma_b$, and $\sigma_R$.

(b) What is the field inside and outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on $q_a$ and $q_b$? *This is problem 2.39 from the book:*

**Problem 2.39**

(a) $\sigma_a = -\frac{q_a}{4\pi a^2}$; $\sigma_b = -\frac{q_b}{4\pi b^2}$; $\sigma_R = \frac{q_a + q_b}{4\pi R^2}$.

(b) $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$, where $\mathbf{r}$ = vector from center of large sphere.

(c) $\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{\mathbf{r}}_a$, $\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b$, where $\mathbf{r}_a$ ($\mathbf{r}_b$) is the vector from center of cavity $a$ ($b$).

(d) Zero.

(e) $\sigma_R$ changes (but not $\sigma_a$ or $\sigma_b$); $\mathbf{E}_{\text{outside}}$ changes (but not $\mathbf{E}_a$ or $\mathbf{E}_b$); force on $q_a$ and $q_b$ still zero.

3. *15 points.* Find the potential inside and outside an infinitely thin spherical shell of radius $R$ that carries a uniform surface charge. Set the reference point at infinity.
*This is Example 2.7:*

**Solution**

From Gauss's law, the field outside is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}},$$

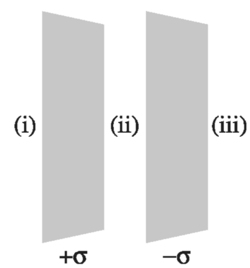
where $q$ is the total charge on the sphere. The field inside is zero. For points outside the sphere ($r > R$),

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\Big|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

To find the potential inside the sphere ($r < R$), we must break the integral into two pieces, using in each region the field that prevails there:

$$V(r) = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{R}\frac{q}{r'^2}\,dr' - \int_{R}^{r}(0)\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\Big|_{\infty}^{R} + 0 = \frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$

4. *15 points.* Consider two infinite parallel planes carrying equal but opposite uniform charge densities $\pm\sigma$.




   (a) Find the electric field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.
   *This is Example 2.6:*

   **Solution**

   The left plate produces a field $(1/2\epsilon_0)\sigma$, which points away from it (Fig. 2.24)—to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $(1/2\epsilon_0)\sigma$, which points *toward* it—to the right in regions (i) and (ii) and to the left in region (iii). The two fields cancel in regions (i) and (iii); they conspire in region (ii). *Conclusion:* The field between the plates is $\sigma/\epsilon_0$, and points to the right; elsewhere it is zero.

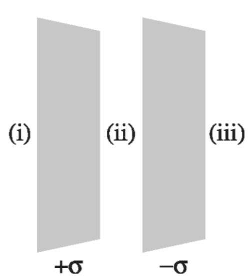



**FIGURE 2.23**

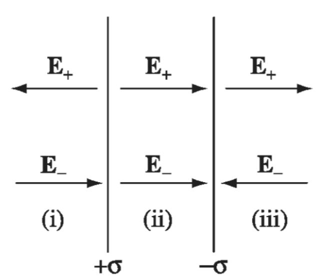



**FIGURE 2.24**

(b) Find the capacitance of a similar configuration, now with finite large metal surfaces of area $A$ held a small distance $d$ apart.

*This is Example 2.11:*

**Solution**

If we put $+Q$ on the top and $-Q$ on the bottom, they will spread out uniformly over the two surfaces, provided the area is reasonably large and the separation small.[13] The surface charge density, then, is $\sigma = Q/A$ on the top plate, and so the field, according to Ex. 2.6, is $(1/\epsilon_0)Q/A$. The potential difference between the plates is therefore

$$V = \frac{Q}{A\epsilon_0}d,$$

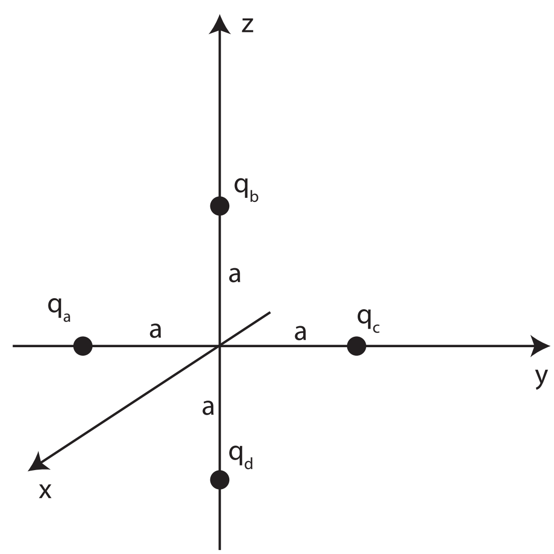
and hence

$$C = \frac{A\epsilon_0}{d}. \qquad (2.54)$$

(c) Let's say this parallel-plate capacitor with finite metal plates has $+Q$ on one plate, $-Q$ on the other. The plates are isolated so the charge $Q$ cannot change. As the plates are pulled apart to double the distance, what happens to the voltage on the plates?

*Answer: The voltage increases. This can be seen from the fact that the capacitance is proportional with $1/d$. The capacitance is also the ratio of $Q/V$, and therefore $V$ must increase to reduce $C$ if $Q$ is constant.*

5. *10 points.* Consider the arrangement of four charges ($q_a, q_b, q_c, q_d$) as depicted on the right. They are in the y-z plane, all at a distance $a$ from the center of the coordinate system.




(a) For $q_a = -2q, q_b = +1q, q_c = -2q, q_d = +3q$, what is (to a good approximation) the electric field at a point P, far away ($x \gg a$) on the x-axis?

(b) Make two 2D sketches (in the x-z plane) in which you compare the electric field of a 'pure dipole' and a physical dipole. The dipoles are oriented along the z-axis, and point in the positive z-direction.

## The End